

Scale Relativistic formulation of non-differentiable mechanics II: The Schrödinger picture

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Abstract

This article is the second in a series of two presenting the Scale Relativistic approach to non-differentiability in mechanics and its relation to quantum mechanics. Here, we show Schrödinger's equation to be a reformulation of Newton's fundamental relation of dynamics as generalized to non-differentiable geometries in the first paper¹. It motivates an alternative interpretation of the other axioms of standard quantum mechanics in a coherent picture. This exercise validates the Scale Relativistic approach and, at the same time, it allows to identify macroscopic chaotic systems considered at resolution time-scales exceeding their horizon of predictability as candidates in which to search for *quantum – like* structuring or behavior.

I. INTRODUCTION

In the first paper¹ of this series of two, we approached continuous but non-differentiable paths (hence scale divergent or commonly fractal) by using a two-term representation in the form $d\mathbf{x}_\pm = \mathbf{v}_\pm dt + d\mathbf{b}_\pm$, in which $d\mathbf{b}_\pm$ is the possibly stochastic residual displacement from a motion with the uniform *usual velocity* \mathbf{v}_\pm over a finite time interval dt . This finite time interval may be regarded as the resolution time-scale used for the inspection of the path. This led to the definition of the complex time-differential operator^{1,14,19}

$$\frac{\hat{d}}{dt} = \frac{1}{2} \left(\frac{d_+}{dt} + \frac{d_-}{dt} \right) - \frac{i}{2} \left(\frac{d_+}{dt} - \frac{d_-}{dt} \right) \quad (1)$$

in which $\frac{d_+}{dt}$ and $\frac{d_-}{dt}$ are finite differentials respectively *after* and *before* the considered point.

In cases where the residual $d\mathbf{b}_\pm$ can be regarded as a Wiener process characterized by $\langle db_{i+} \cdot db_{i-} \rangle = 0$, and $\langle db_{i\pm} \cdot db_{j\pm} \rangle = 2\mathcal{D}\delta_{i,j}dt$, with \mathcal{D} akin to a diffusion coefficient, the complex time-differential operator takes the form^{1,14,19}

$$\frac{\hat{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V} \cdot \nabla - i\mathcal{D}\Delta \quad (2)$$

in which $\mathcal{V} = \mathbf{V} + i\mathbf{U}$ is the complex velocity, with $\mathbf{V} = \frac{1}{2}(\mathbf{v}_+ + \mathbf{v}_-)$ corresponding to the classical velocity while $\mathbf{U} = \frac{1}{2}(\mathbf{v}_+ - \mathbf{v}_-)$ is the *kink velocity* measuring the discontinuity of the velocity, both at the considered point and resolution time-scale.

Applying the stationary action principle with the now complex action $\mathcal{S} = \int_0^t \mathcal{L}(x, \mathcal{V}, t)dt$, we recovered the Euler-Lagrange equations with the time derivative and velocity respectively replaced by $\frac{\hat{d}}{dt}$ and \mathcal{V} . In the special case of the Lagrange function $\mathcal{L} = \frac{1}{2}m\mathcal{V}^2 - \Phi$ where Φ is a real potential energy, we obtain a similarly generalized version of Newton's fundamental relation of dynamics^{1,14,19}:

$$m \frac{\hat{d}}{dt} \mathcal{V} = -\nabla \Phi \quad (3)$$

In Section II of this paper, we are going to see how this generalized fundamental relation of dynamics can be rewritten in the form of a Schrödinger equation. This motivates the interpretation of the system of axioms of quantum mechanics in terms of non-differentiable paths presented in Section III. In Section IV, we then consider how the Scale Relativistic approach to quantum mechanics can be transposed to complex or chaotic systems. This justifies the ongoing search for *quantum-like* signatures in the structures and dynamics of such systems. Section V then provides a summary and discussion.

II. RECOVERING WAVE-FUNCTIONS AND SCHRÖDINGER'S EQUATION

The complex action $\mathcal{S}^{14,19}$ can be re-expressed logarithmically in terms of a function ψ with $\mathcal{S} = -i\mathcal{S}_0 \ln(\psi/\psi_0)$, in which ψ_0 and \mathcal{S}_0 are introduced for dimensional reasons. This can be used to express the complex velocity \mathcal{V} by using the canonical momentum $\mathcal{P} = m\mathcal{V} = \nabla\mathcal{S}$ or $\mathcal{V} = -i\frac{\mathcal{S}_0}{m}\nabla \ln(\psi/\psi_0)$. We see that ψ_0 cancels out of the expression of \mathcal{V} . For this reason and in order to lighten notations, we will start writing $\ln \psi$ in place of $\ln(\psi/\psi_0)$. This expression of \mathcal{V} can be used together with the complex differential operator (Equation 2) in the generalized fundamental relation of dynamics (Equation 3), where we introduce $\eta = \frac{\mathcal{S}_0}{2m\mathcal{D}}$:

$$2im\mathcal{D}\eta \left[\frac{\partial}{\partial t} (\nabla \ln \psi) - i\mathcal{D} (2\eta(\nabla \ln \psi \nabla)(\nabla \ln \psi) + \Delta(\nabla \ln \psi)) \right] = \nabla \Phi.$$

The identity^{14,19} demonstrated in Appendix A can be applied directly to obtain:

$$2im\mathcal{D}\eta \left[\frac{\partial}{\partial t} (\nabla \ln \psi) - i\frac{\mathcal{D}}{\eta} \nabla \left(\frac{\Delta \psi^\eta}{\psi^\eta} \right) \right] = \nabla \Phi.$$

Since all the terms are gradients, this can be integrated to

$$2im\mathcal{D}\eta \frac{\partial \ln \psi}{\partial t} = -2m\mathcal{D}^2 \left(\frac{\Delta \psi^\eta}{\psi^\eta} \right) + \Phi + \Phi_0$$

where the integration constant Φ_0 can always be absorbed in the choice of the origin of the energy scale so we do not carry it further. Developing the Laplacian and using $\mathcal{P} = -i\mathcal{S}_0 \nabla \ln \psi = -2im\mathcal{D}\eta \nabla \ln \psi$, we obtain

$$2im\mathcal{D}\eta \frac{\partial \psi}{\partial t} = \frac{\eta - 1}{\eta} \frac{\mathcal{P}^2}{2m} \psi - 2m\mathcal{D}^2 \eta \Delta \psi + \Phi \psi.$$

If we now choose $\eta = 1$, which corresponds to setting the value of the reference action $\mathcal{S}_0 = 2m\mathcal{D}$, we finally obtain Schrödinger's equation in which \hbar is replaced with $2m\mathcal{D}$:

$$2im\mathcal{D} \frac{\partial \psi}{\partial t} = -2m\mathcal{D}^2 \Delta \psi + \Phi \psi$$

This result is similar to that obtained by Edward Nelson in his 1966 article entitled "Derivation of the Schrödinger Equation from Newtonian Mechanics"¹³, in which he "examined the hypothesis that every particle of mass m is subject to a Brownian motion with diffusion coefficient $\hbar/2m$ and no friction. The influence of an external field was expressed by means of Newton's law $F = ma, \dots$."

However, the similarity is only superficial. Nelson concluded that "*the hypothesis leads in a natural way to Schrödinger's equation, but the physical interpretation is entirely classical*"¹³. Indeed, the diffusive process was postulated to be at play at some *sub-quantum* level, making it a hidden variable theory even if "*the additional information which stochastic mechanics seems to provide, such as continuous trajectories, is useless, because it is not accessible to experimental verification*"¹³. In the Scale Relativity approach^{14,19} followed here, Schrödinger's equation does not result from any additional hypothesis. Instead, it results from the relaxation of the usually implicit hypothesis of differentiability for the space coordinates. Using the resolution-scale specific two-term path representation, we have seen that this relaxation corresponds to considering resolution-scales as additional relative attributes of reference frames, which is the central idea of Scale Relativity. The identification of the doubling of the velocity field in the two-term path representation¹ led to the appearance of complex numbers^{14,19} and is in itself a definite departure from any *trajectory* based *classical interpretation*. With this, the generalized Newton relation and the equivalent Schrödinger equation take form under the specific restriction to paths of fractal dimension 2 corresponding to Wiener processes. Schrödinger's equation then appears as just one in a family of generally more intricate equations for stochastic processes with different statistics.

Also, Nelson described quantum particles as having "*continuous trajectories and the wave function is not a complete description of the state*"¹³. When discussing the simulation presented in Figure 3 of the first paper of this series, we already commented on the fact that, in the limit of infinitesimal time steps, the position of the particle as a definite property becomes a meaningless concept, which has to be replaced with a probabilistic description. The consideration of *one specific path* being followed by a particle then is a misconception. Starting from the fundamental relation of dynamics generalized to non-differentiable paths, we arrive to Schrödinger's equation with the *wave function* $\psi(\mathbf{x}, t)$ identified to an exponential re-expression of the action. If the statistics of the stochastic component of the path is preserved all the way down to infinitesimal resolution time-scales, the state of the system can no longer be specified by coordinates values. With the disappearance of a specific path actually followed by the system, one must recognize the function ψ as *completely* specifying the state of the system.

Schrödinger's equation as a prescription for the time evolution of the state of the system is only one of the axioms founding quantum mechanics. In the following section, we discuss

the interpretation of the other axioms in the Scale Relativistic approach.

III. THE AXIOMS OF QUANTUM MECHANICS

Standard quantum mechanics is built up from the enunciation of a number of mathematical postulates⁵, which are generally not considered to derive from any more fundamental principles and are justified by the predictive power of their application. The first of these axioms specifies that the state of a system can be represented by a state-vector $|\psi\rangle$ belonging to a complex vectorial state-space specific to the considered system. Another postulate, the Schrödinger postulate, prescribes the time evolution of the state of a system to be driven by Schrödinger's equation $i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$, where \hat{H} is the observable operator associated with the system total energy. The derivation of Schrödinger's equation in position representation (Section II) identifies the wave-function $\psi(t, \mathbf{x}) = \langle \mathbf{x} | \psi(t) \rangle$ (where $|\mathbf{x}\rangle$ is the state of definite position \mathbf{x}) to an exponential expression of the action \mathcal{S} , now a complex quantity in direct consequence of our consideration of non-differentiable paths. We already argued that the inclusion of non-differentiable paths amounts to a departure from a trajectory based description of the state of the system, leaving the wave function as the complete description of the state of the system. The time independent Schrödinger equation obtained by separation of variables is an eigenvalue equation. Its solutions constitute a vectorial space, which establishes the first postulate. In this section we discuss the other postulates from the Scale Relativistic point of view¹⁸.

A. Observables as state-space operators

The postulate on observables states that any physical quantity \mathcal{O} that can be measured is associated with an hermitian operator \hat{O} acting on the state space. Such an operator is known as an observable. The measurement of a physical quantity can only yield one of the observable's eigenvalues a_i as a result.

In the course of the derivation of Schrödinger's equation, we have already identified the expression for the complex momentum $\mathcal{P}(\mathbf{x}) = \nabla \mathcal{S} = -i\mathcal{S}_0 \nabla \ln \psi$ which can be rewritten as $\mathcal{P}(\mathbf{x})\psi = -i\mathcal{S}_0 \nabla \psi = \hat{P}\psi$. Similarly, for the energy we have $\mathcal{E}(\mathbf{x}) = \frac{\partial \mathcal{S}}{\partial t} = i\mathcal{S}_0 \frac{\partial \ln \psi}{\partial t}$ or $\mathcal{E}(\mathbf{x})\psi = i\mathcal{S}_0 \frac{\partial \psi}{\partial t} = \hat{E}\psi$. The correspondence between the physical quantity \mathcal{P} or \mathcal{E} and

the respective linear operator \hat{P} or \hat{E} acting on the state space is actually replaced by an equality. In both cases, the operator is found to be hermitian. When considering a state of definite momentum or energy, we may require \mathcal{P} or \mathcal{E} to be independent of \mathbf{x} . This then implies that the only possible values of a definite momentum or energy are the solutions of the usual eigenvalue equations.

More generally, in classical mechanics, any physical quantity characterizing the state of the system can be expressed as the result of some local operation on the classical action considered as a function of time and the system's coordinates. This may be generalized to the complex action \mathcal{S} and, alternatively, we may consider the wave-function $\psi = \psi_0 e^{i\mathcal{S}/\hbar}$ as a starting point. Then, any physical quantity \mathcal{O} characterizing the state of the system can be expressed as the result of some local operation on the wave function ψ . Anticipating the wave function $\psi(t, \mathbf{x})$ as the complex probability amplitude of Born's postulate to be discussed in the next subsection (III B), implies $\mathcal{O}(\mathbf{x})$ to be insensitive of the normalization and global phase of the wave function. This justifies the writing $\hat{O}\psi = \mathcal{O}(\mathbf{x})\psi$ with \hat{O} a linear operator. When considering a state of definite \mathcal{O} , we may require $\mathcal{O}(\mathbf{x})$ to be independent of \mathbf{x} and obtain an eigenvalue equation, which, as above, determines the only possible definite values of \mathcal{O} . The nature of the measurement process will be clarified in the discussion of von Neumann's postulate in subsection III C. We however already see how regarding measurements outcomes as definite values of \mathcal{O} implies they can only be eigenvalues of \hat{O} . In turn, the fact measurement outcomes are real quantities implies the observables \hat{O} to be hermitian operators.

B. Born's postulate

Born's postulate states that, for a system in a normalized state $|\psi\rangle$ (so that $\langle\psi|\psi\rangle = 1$), the measurement of a quantity \mathcal{O} yields one of the eigenvalues o_i of the associated observable operator \hat{O} , with a chance probability given by the squared magnitude of the component of $|\psi\rangle$ in the sub-state-space corresponding to the observable eigenvalue o_i . In the case of position measurements, this postulate means that, in terms of the wave function $\psi(t, \mathbf{x})$, the probability density of finding the particle in \mathbf{x} is given by $|\langle\mathbf{x}|\psi(t)\rangle|^2 = |\psi(t, \mathbf{x})|^2$.

Writing the wave function as $\psi = \sqrt{\rho} e^{i\chi}$ in Schrödinger's equation established in Section II, with both ρ and χ real, and separating the real and imaginary parts, result in the

Madelung¹⁰ equations (See appendix B for details):

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}) \quad (4)$$

$$\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right) \mathbf{V} = -\frac{\nabla(\Phi + \mathcal{Q})}{m} \quad (5)$$

Equation 4 is a continuity equation in which $\rho = \psi^* \psi$ plays the role of the fluid density with a velocity field $\mathbf{V} = \frac{\hbar}{m} \nabla \chi$ (See Appendix B).

Equation 5 is Euler's equation of fluid dynamics with the additional gradient of the quantum potential $\mathcal{Q} = -2m\mathcal{D}^2 \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$ (See Appendix B), which appears to be entirely responsible for the quantum behavior. In the Scale Relativistic interpretation, the quantum potential is a manifestation of the fractal nature of the paths from which it derives. This is quite comparable to the situation in general relativity which reveals the gravitational potential as a manifestation of the curved nature of space-time²².

We already commented on the fact that abandoning the hypothesis of path differentiability results in the loss of path discernibility. If it were meaningful, following one path would amount to following them all. This restricts the consideration of position to a probabilistic description. Schrödinger's equation is now rewritten as a fluid dynamics equation. This naturally leads to identifying the fluid density $\rho = \psi \psi^*$ to the normalized density of indiscernible contributing paths. The normalized path density then sets the probability density of position measurement outcomes. Indeed, the indiscernibility of the paths implies they are equally likely to be expressed in the measurement outcome. The time evolution of the system appears as a bundle of an infinity of indiscernable paths. The cross section of the bundle at a given time t constitutes the state of the system and is described by the wave function $\psi(t, \mathbf{x})$ solution of Schrödinger's equation. The normalized path density $\psi(t, \mathbf{x})\psi^*(t, \mathbf{x})$ of the bundle at time t , is the probability density for finding the particle in \mathbf{x} .

This establishes Born's postulate in the position representation. The wave function $\psi = \langle \mathbf{x} | \psi \rangle$ is just the position representation of an abstract state vector $|\psi\rangle$. Observables, being Hermitian, the state can be represented in terms of the eigenstates of any complete set of commuting observables. That is to say, unitary transformations can be used to go from one representation to another. This generalizes Born's postulate in position representation to any representation. The measurement of \mathcal{O} gives o_i with the probability $|\langle o_i | \psi \rangle|^2$ where $|o_i\rangle$ is a state of definite value $\mathcal{O} = o_i$, the eigenvector of $\hat{\mathcal{O}}$ associated with the eigenvalue o_i .

C. von Neumann's postulate

The *wave function collapse* or von Neumann postulate specifies that, immediately after a measurement of \mathcal{O} yielding o_i , the system is in the state given by the projection of the initial state onto the eigen-sub-state-space corresponding to the eigenvalue o_i of the observable \hat{O} corresponding to \mathcal{O} .

Given the above interpretation of Born's postulate with the wave functions describing the set of the non-differentiable and indiscernible paths, a measurement of \mathcal{O} can naturally be envisioned as the selection of a bundle of indiscernible paths corresponding to the measurement outcome o_i , eigenvalue of \hat{O} . The associated eigenstate vector $|o_i\rangle$ represents the bundle of paths that is selected and for which \mathcal{O} has the definite value o_i . It might be worth stressing again that the system should not be thought as following a specific path in the bundle. This would be a misconception precisely because of the indiscernible character of the paths. The identification of the path bundle to the state of the system itself with the measurement amounting to a path bundle selection implies that the state of the system immediately after the measurement yielding o_i precisely is $|o_i\rangle$. Following the initial measurement, after a time short enough for the time evolution of the selected bundle of paths to be negligible, a second measurement of \mathcal{O} does not result in any further alteration of the paths bundle. The state vector remains $|o_i\rangle$ and the second measurement results in the same outcome o_i with a unit probability. This interpretation of von Neumann postulate follows the picture given for Born's postulate. The relative number of paths in the bundle $|o_i\rangle$ contributing to a state $|\psi\rangle$ is $|\langle o_i|\psi\rangle|^2$, which sets the chance probability for the first measurement of \mathcal{O} to yield o_i .

D. Systems with more than one particle

The previous subsection completes the Scale Relativistic interpretation of the postulates of standard quantum mechanics. We went through the derivation of Schrödinger's equation for one particle in the usual three-dimensional physical space plus time. The exact same derivation can be carried out for an arbitrary number of dimensions. In particular, when considering a system composed of N particles, we would obtain the same Schrödinger equation in a $3N$ dimensional physical space plus time, with, possibly, a different mass for each

particle. The potential energy term may then depend on the relative coordinates of different particles to account for their mutual interactions.

In quantum mechanics, one talks about entanglement¹² when two or more particles emerge from a mutual interaction in such a way one can only talk about the quantum state of the system as a whole and not about the quantum states of the constituents considered individually. The resulting correlation between the outcomes of the measurements of individual particles in the system is a major characteristic aspect of standard quantum mechanics²¹.

In the Scale Relativity interpretation of quantum mechanics, an entangled state $|\psi_N\rangle$ of N particles would correspond to a bundle of non-differentiable and indiscernible paths in the $3N$ dimensional physical space. The bundle may branch out in a number of sub-bundles corresponding to the various configurations in which the system might be found during a subsequent measurement of some of its constituent particles. Following the above interpretation of Born's and von Neumann's postulates, the statistics of the paths in the sub-bundles corresponds to the probabilities of the various possible outcomes of measurements of individual particles. We may stress again that the system of N particles should not be thought as following one specific path. Instead, the state of the system is to be identified to the entire bundle of paths described by the state vector $|\psi_N\rangle$. The measurement of some particles then selects a part of the bundle and may immediately provide information about other particles, not involved in the measurement, because of the specific structure of the bundle which implements the appropriate correlation between the different parameters that may be measured in an experiment probing the entanglement.

It appears that the early implementation of Scale Relativity in the development of point mechanics with time as an absolute external parameter lands onto a coherent foundation of standard quantum mechanics^{14,19}. This should at least be regarded as a validation of the Scale Relativity proposal. More has been done since with, in particular, a Scale Relativity approach to motion relativistic quantum mechanics⁴ and also to gauge theories¹⁶. Here, in the next section, we follow a different direction and consider the possibility that some macroscopic systems, which can be described as evolving along non-differentiable paths, could fall under a standard *quantum-like* description.

IV. CHAOS STRUCTURED BY A QUANTUM-LIKE MECHANICS

The above Scale Relativistic foundation of standard quantum mechanics does not result in any aspect different from other approaches that could be tested experimentally. It emerges from the consideration of non-differentiable stochastic paths described as Wiener processes all the way down to infinitesimal resolution-scales. It should however be noted that the derivations of the generalized fundamental relation of dynamics¹ and the equivalent Schrödinger equation do not depend in any way on the assumption that the fractal nature of the paths is preserved uniformly all the way down to infinitesimal resolution time-scales. If there is a resolution time-scale below which the paths lose their stochastic component, becoming differentiable and discernible again, Schrödinger's equation still holds and the other postulates are still applicable as long as the system is considered at sufficiently long resolution time-scales for the details of the evolution to be reducible to the statistical description of an effectively stochastic Wiener process.

Chaotic systems are characterized by a high sensitivity to initial conditions⁷. This is generally described in terms of the rate at which two infinitesimally close trajectories move apart from each other, which defines the Lyapunov time-scales over which the chaotic nature of the dynamic system expresses itself. While the system may be evolving in a deterministic way, predictions of the evolution of the system over time intervals much exceeding the Lyapunov times are not reliable. This is often referred to as a predictability horizon. When the system is observed with resolution time-scales well in excess of the predictability horizon, the successive configurations appear random and uncorrelated. They sample an ensemble following a probability density map possibly evolving with time. The observation of the system with finer resolution-scales in attempts to better characterize an elusive trajectory keeps revealing new structures until one reaches the Lyapunov time-scale where the non-differentiable paths condensate in a differentiable trajectory and predictability is recovered. So as long as the system is considered at resolution time-scales well exceeding the Lyapunov time, the developments that led us to a Scale Relativistic interpretation of quantum mechanics should be applicable.

This has the intriguing consequence that the postulates of quantum mechanics may be applicable to macroscopic complex and/or chaotic systems outside the realm of standard quantum mechanics. The Planck constant in the Schrödinger equation would then be re-

placed by some different value $2m\mathcal{D}$ to be identified and which could be system specific. Interestingly, the mass m of the particle cancels out in the expression of the generalized de Broglie length $\lambda = \frac{2\mathcal{D}}{|\mathbf{v}|}$ and the velocity may then be expected to play a role similar to that of the momentum in standard quantum mechanics. Other than this, the main difference from standard quantum mechanics would lie in the fact that, at fine resolution-scale, a deterministic predictable behavior is recovered. So, in classical systems considered beyond their predictability horizon where classical mechanics fails and no alternative theory is currently available, a *quantum-like* mechanics may be applicable to provide some account for their often rich structuring.

This justifies the search of *quantum-like* features complex and chaotic or stochastic systems. Virtually dissipation-less Keplerian Astrophysical systems constitute a domain of predilection for such searches. In fact the possibility *quantum-like* structuring could be found in Keplerian gravitational system was considered just a few years after the publication of Schrödinger's equation with an analysis of the orbits of the major objects of the Solar system as well as the orbits of their satellites^{3,11,20}. These analyses were performed again in more details in the Scale Relativistic context^{8,15} and even included an account for the masses of the major objects of the Solar system following a *quantum-like* hydrogenoid orbital profile of the distribution of coalescing primordial planetesimals. Similar analyses were performed for Kuiper belt objects in the Solar System¹⁹, extra-solar planetary systems¹⁷, binary stars, pairs of galaxies and others¹⁹. All are suggestive such a *quantum-like* mechanics is at play in the structuring of these systems following multiples and submultiples of a seemingly universal velocity. While the compilation of these results is already striking, it would be highly desirable to achieve laboratory based experiment so the *quantum-like* dynamics with the emergence of characteristic velocities in the quantization can be tested in a controlled environment.

V. CONCLUSION

The proposal of Scale Relativity is to include resolution-scales as additional relative parameters defining reference frames with respect to each others. In the first paper¹ of this series, we considered non-differentiable paths as an implementation of Scale Relativity for the description of motion with time as an external parameter. The time interval between

kinks corresponds to a path inspection resolution time-scale. Restricting ourselves to paths with a stochastic component corresponding to a Wiener process, we obtained a generalized form of Newton's fundamental relation of dynamics (Equation 3) in terms of the complex time-differential operator $\frac{\hat{d}}{dt}$ (Equation 1) and the corresponding complex velocity $\mathcal{V} = \frac{\hat{d}\mathbf{x}}{dt}$. Guided by numerical simulations, we understood that such a dynamics corresponds to an abandonment of the notion of trajectory, which has to be replaced by an exclusively probabilistic consideration of position.

In section II of this paper, starting from the generalized equation of dynamics (Equation 3) and expressing the complex velocity \mathcal{V} in terms of ψ an exponential expression of the complex action \mathcal{S} , we obtained a Schrödinger's equation with \hbar replaced by $2m\mathcal{D}$. We stressed again that, because of their non-differentiability, the paths are indiscernible in such a way it would be a misconception to envision a specific one to be followed by the system. Instead, the wave function ψ must be recognized as completely specifying the state of the system as long as the fractal nature of the paths is preserved all the way to infinitesimal resolution-scales. As such, the wave function $\psi(t, \mathbf{x})$ can be regarded as a description of the cross section of a bundle of an infinity of non-differentiable Wiener paths at time t .

With this, in Section III, we proceeded to a coherent Scale Relativistic interpretation of each of the postulates founding standard quantum mechanics. In particular, using Madelung's equations, the wave function was identified to the bundle's normalized path density amplitude which we assimilated to the probability amplitude of Born's postulate.

The establishment of Schrödinger's equation and the coherent interpretation of the postulates of quantum mechanics is a major success of the Scale Relativity method. This validation was continued with the application to relativistic⁴ and gauge¹⁶ quantum theories. Quantum mechanics is characterized by the fact it includes a dependance on resolution-scales as expressed most clearly by Heisenberg uncertainty relations. In hindsight, it is not surprising that Scale Relativity provides a natural and less axiomatic accommodation of quantum mechanics as the consideration for resolution-scale dependance is included from the very start by the extension of the relativity principle to scaling laws. This success of the implementation of the Scale Relativity principle may bring the question of the quantization of the gravitational interaction under a different light. The gravitational curvature of space-time at *large* scales may be seen as giving way to a dense structuring at small scales with non-differentiable and indiscernable geodesics in the quantum domain. The Scale Rel-

ativity method could provide an avenue for revealing quantum mechanics and the general relativistic description of gravitation, as being in continuation of each other.

In Section IV, we remarked that the preceding developments did not depend on the fractal nature of the considered paths to be uniformly preserved all the way down to infinitesimal resolution-scales. In particular, the postulates of quantum mechanics should remain applicable even if the paths loose their fractal character below some characteristic resolution-scale. The only requirement is that at the considered resolution-scale, the system's evolution is appropriately described by a Wiener process. This opens the possibility for the structuring of some complex and/or chaotic systems to be structured according to the laws of a *quantum-like* mechanics provided the systems are considered over resolution time-scales exceeding their predictability horizon. This is precisely the domain in which classical mechanics loses its predictive power leaving probabilistic descriptions as the only valid approach while there is no currently accepted general tool or theory allowing for the prediction of probability densities. The observations of various astrophysical systems are indicating a *quantum-like* mechanics is at play in their structuring. Here again, in hindsight it would not be surprising that Scale Relativity could provide a fruitful insight in complex/chaotic systems as their behavior is generally characterized by couplings across broad ranges of scales. Additionally, it should be noted that the Schrödinger equation was obtained in the very restrictive case of paths with a Wiener process as their stochastic component. This corresponds to paths whose stochastic component is memoryless or Markovian. The consideration of different statistics would result in different dynamics, which maybe able to provide an account for structuring occurring in a broader range of natural complex systems.

To summarize, it appears the Scale Relativity principle provides a new approach to the foundation of quantum mechanics and may provide an effective method of theoretical research in the microphysical world. At the same time, it seems to provide an avenue to extend the reach of fundamental physics methods to integrate complex and chaotic system, the approach to which are otherwise restricted to effective and phenomenological descriptions. The program is ambitious and opens up on many possibilities of experimental, observational and theoretical developments in physics as well as in interdisciplinary fields.

Appendix A: A useful identity

Lets look at $(\nabla \ln \psi)^2 + \Delta \ln \psi = \partial_i \ln \psi \partial_i \ln \psi + \partial_i \partial_i \ln \psi = \frac{\partial_i \psi \partial_i \psi}{\psi^2} + \partial_i \frac{\partial_i \psi}{\psi}$ where $i = \{x, y, z\}$ with summation over repeated indices. $(\nabla \ln \psi)^2 + \Delta \ln \psi = \frac{\partial_i f \partial_i \psi}{\psi^2} + \frac{\psi \partial_i \partial_i \psi - \partial_i \psi \partial_i \psi}{\psi^2} = \frac{\Delta \psi}{\psi}$. We can then take the gradient:

$$\nabla(\nabla \ln \psi)^2 + \nabla \Delta \ln \psi = \nabla \left(\frac{\Delta \psi}{\psi} \right).$$

We now concentrate on the first term on the left hand side where we note $f = \ln \psi$: $\nabla(\nabla f)^2 = \partial_i \partial_j f \partial_j f = 2 \partial_j f \partial_j \partial_i f$ so we get

$$\nabla(\nabla f)^2 = 2(\nabla f \cdot \nabla) \nabla f.$$

So in total, using the fact that $\nabla \Delta = \Delta \nabla$, we can write:

$$2(\nabla \ln \psi \cdot \nabla) \nabla \ln \psi + \Delta(\nabla \ln \psi) = \nabla \left(\frac{\Delta \psi}{\psi} \right).$$

Applying this to ψ^η and dividing by η we obtain:

$$2\eta(\nabla \ln \psi \cdot \nabla) \nabla \ln \psi + \Delta(\nabla \ln \psi) = \frac{1}{\eta} \nabla \left(\frac{\Delta \psi^\eta}{\psi^\eta} \right).$$

Appendix B: Madelung equations

In position representation, the kinetic energy corresponds to the operator $\hat{T} = -\frac{\mathcal{S}_0}{2m} \Delta$ so that

$$\hat{T}\psi = -\frac{\mathcal{S}_0^2}{2m} \Delta (\sqrt{\rho} e^{i\chi}) \tag{B1}$$

$$= -\frac{\mathcal{S}_0^2}{2m} e^{i\chi} (\Delta \sqrt{\rho} + 2i \nabla \sqrt{\rho} \nabla \chi - \sqrt{\rho} (\nabla \chi)^2 + i \sqrt{\rho} \Delta \chi) \tag{B2}$$

Using $e^{i\chi} = \frac{\psi}{\sqrt{\rho}}$, simplifying by ψ and rearranging a little, we obtain an expression for the kinetic energy:

$$\mathcal{T} = \frac{\mathcal{S}_0^2}{2m} (\nabla \chi)^2 - \frac{\mathcal{S}_0^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} - i \frac{\mathcal{S}_0^2}{2m \rho} \nabla (\rho \nabla \chi) \tag{B3}$$

$$= \frac{1}{2} m \mathbf{V}^2 + \mathcal{Q} - i \frac{\mathcal{S}_0}{2\rho} \nabla (\rho \mathbf{V}) \tag{B4}$$

In the second line we have introduced $\mathbf{V} = \frac{\mathcal{S}_0}{m} \nabla \chi$, the classical velocity field, with which the first term appears as the classical kinetic energy. It is worth noting that if we write

$\chi = \frac{\mathcal{S}'}{2m\mathcal{D}}$, with \mathcal{S}' the real part of the action, the expression of the velocity field corresponds to the relation $\mathbf{V} = \frac{\nabla \mathcal{S}'}{m}$. As the gradient of the real part of the action equals the gradient of the classical action, this makes \mathbf{V} clearly appear as a field of *usual velocity*.

We also introduced the potential $\mathcal{Q} = -\frac{\mathcal{S}_0^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$, the gradient of which was described by Erwin Madelung in his 1927 paper as "an internal force of the continuum"¹⁰. The energy \mathcal{Q} is now referred to as the *quantum potential*, a denomination due to Bohm in 1952². Despite this denomination and the role of \mathcal{Q} in Madelung's equation, we see that the quantum potential truly is of a kinetic nature⁹.

We can proceed in the same way with the energy $\hat{E}\psi = i\mathcal{S}_0 \frac{d}{dt} (\sqrt{\rho} e^{i\chi})$, which leads to $\mathcal{E} = -\mathcal{S}_0 \frac{d\chi}{dt} + i\frac{\mathcal{S}_0}{2\rho} \frac{d\rho}{dt}$.

With the inclusion of the potential energy Φ , the imaginary part of the equation $\mathcal{E} = \mathcal{T} + \Phi$ gives the continuity equation (Equation 4) while the gradient of the real part gives Euler's fluid dynamics equation (Equation 5).

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